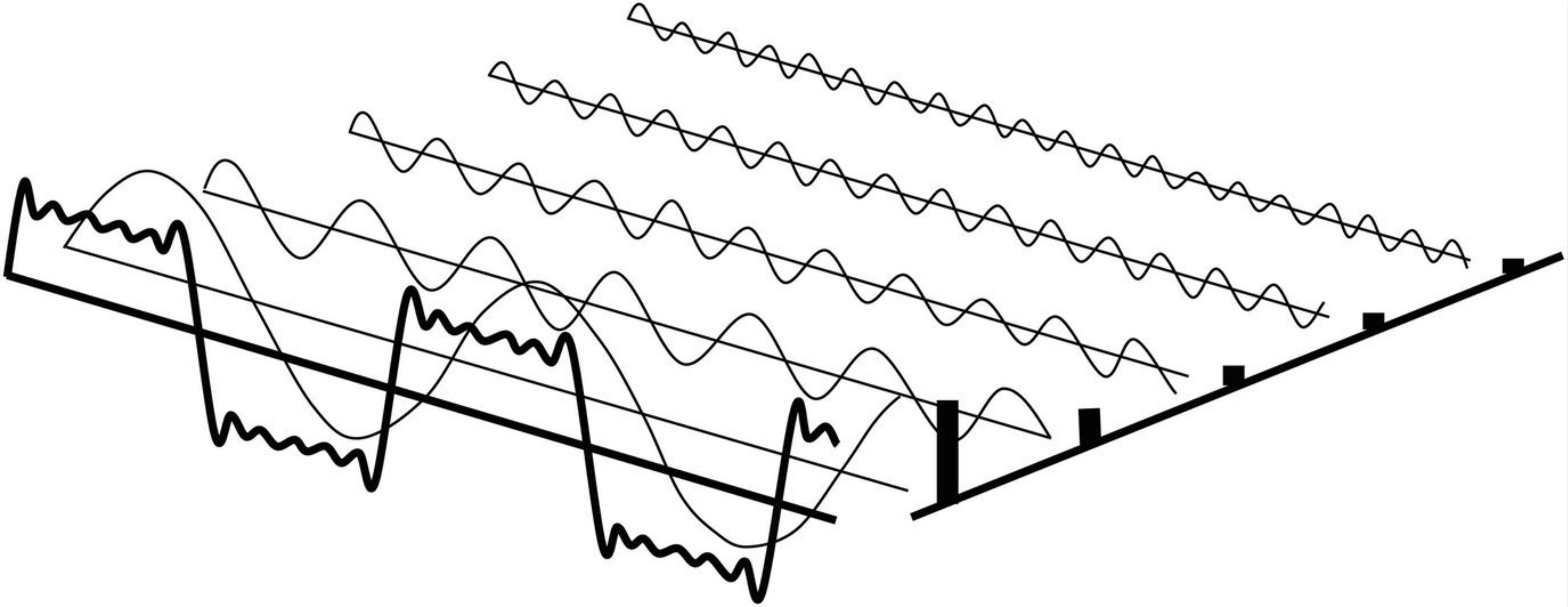
Paper: PHS-A-CC-3-5-TH Credit:4

1. Fourier Series 10 Lectures

(a) Periodic functions. Orthogonality of sine and cosine functions, Dirichlet Conditions (Statement only). Expansion of periodic functions in a series of sine and cosine functions and determination of Fourier coefficients. Complex representation of Fourier series. Expansion of functions with arbitrary period. Expansion of non-periodic functions over an interval. Even and odd functions and their Fourier expansions. Applications. Summing of Infinite Series. Term-by-Term differentiation and integration of Fourier Series. Parseval Identity.

Fourier Series



Fourier Series

http://www.gap-system.org/~history/PictDisplay/Fourier.html

Joseph Fourier
1768-1830

"In 1822, Joseph Fourier, a French mathematician, discovered that sinusoidal waves can be used as simple building blocks to describe and approximate any periodic waveform including square waves. Fourier used it as an analytical tool in the study of waves and heat flow. It is frequently used in signal processing and the statistical analysis of time series."

http://en.wikipedia.org/wiki/Sine_wave

Introduction to Fourier Series

- It is named after French mathematician and physicist ‘Jacques Fourier’ (1768-1830)’
- A series expansion of a function in terms of trigonometric functions cos mx and sin nx is called Fourier series.
- Many functions including some discontinuous periodic functions can be written in a Fourier series
- It has wide applications in solving some ordinary and partial differential equations.

Orthogonal functions

The set of functions

$$\left\{1, \cos\frac{\pi x}{\ell}, \cos\frac{2\pi x}{\ell}, \ldots, \sin\frac{\pi x}{\ell}, \sin\frac{2\pi x}{\ell}\ldots\right\}$$

is orthogonal in $[-\ell, \ell]$ since

- $\displaystyle\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}dx = \int_{-\ell}^{\ell} \sin\frac{m\pi x}{\ell}dx = 0.$

- $\displaystyle\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx = \begin{cases} 0 & \text{if } m \neq n, \\ \ell & \text{if } m = n. \end{cases}$

Mathematical expression

- $$\int_{-\ell}^{\ell} \sin\frac{m\pi x}{\ell} \sin\frac{n\pi x}{\ell} dx = \begin{cases} 0 & \text{if } m \neq n, \\ \ell & \text{if } m = n. \end{cases}$$
- $$\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell} \sin\frac{n\pi x}{\ell} dx = 0, \ \forall \ m, \ n.$$

Expansion of periodic function in Fourier series

Now, let $f(x)$ be a periodic function of period 2ℓ defined on $[-\ell, \ell]$. Assume that it can be expressed as a linear combination of trigonometric functions $\cos mx$ and $\sin nx$. That is,

$$f(x) = \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}\right]. \qquad (1)$$

Integrate (1) both sides from $-\ell$ to ℓ, we have

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx.$$

Fourier coefficients

Now, multiply (1) by $\cos\dfrac{n\pi x}{\ell}$ and then integrate from $-\ell$ to ℓ, we obtain

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx,\ \ n = 1, 2, 3, \ldots$$

Similarly, multiply (1) by $\sin\dfrac{n\pi x}{\ell}$ and then integrate from $-\ell$ to ℓ, we get

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx,\ \ n = 1, 2, 3, \ldots$$

These formulae of $a_0,\ a_n$ and b_n, above are called Euler's formulae.

Problem 1

Find the Fourier series expansion of the following periodic function of period 2π

$$f(x) = \begin{cases} \pi + x & \text{if } -\pi < x < 0, \\ 0 & \text{if } 0 \leq x < \pi. \end{cases}$$

Problem 2

- Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$, $f(x + 2\pi) = f(x)$.

  Hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.$$

Convergence of Fourier Series for continuous functions

If a periodic function $f(x)$ with period 2ℓ is continuous in $[-\ell, \ell]$ and has continuous first and second derivatives at each point in that interval, then the Fourier series

$$\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}\right],$$

of $f(x)$ is convergent.

Formula

The Fourier coefficients $a_0,\ a_n$ and b_n are given by

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx,$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx,$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx.$$

Calculation

Now,

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx$$

$$= \frac{1}{\ell}\left[\left(f(x)\frac{\sin(n\pi x)/\ell}{(n\pi)/\ell}\right)_{-\ell}^{\ell} - \frac{\ell}{n\pi}\int_{-\ell}^{\ell} f'(x)\sin\frac{n\pi x}{\ell}dx\right]$$

$$= -\frac{1}{n\pi}\int_{-\ell}^{\ell} f'(x)\sin\frac{n\pi x}{\ell}dx, \quad [\text{since } \sin n\pi = 0].$$

Calculation

After integration by parts again, we get

$$a_n = -\frac{1}{n\pi}\Big[\Big(- f'(x)\frac{\cos(n\pi x)/\ell}{(n\pi)/\ell}\Big)_{-\ell}^{\ell} + \frac{\ell}{n\pi}\int_{-\ell}^{\ell} f''(x)\cos\frac{n\pi x}{\ell}dx\Big]$$

The first term on the right side is zero because of periodicity and continuity of $f'(x)$. Since $f''(x)$ is continuous in the interval $[-\ell, \ell]$, therefore

$$|f''(x)| < M$$

for an appropriate constant M. Also, $|\cos\frac{n\pi x}{\ell}| \leq 1$.

Calculation

Therefore,

$$|a_n| = \frac{\ell}{n^2\pi^2}\left|\int_{-\ell}^{\ell} f''(x)\cos\frac{n\pi x}{\ell}dx\right|$$

$$< \frac{\ell}{n^2\pi^2}\left|\int_{-\ell}^{\ell} Mdx\right.$$

$$= \frac{2M\ell^2}{n^2\pi^2}$$

Similarly,

$$|b_n| < \frac{2M\ell^2}{n^2\pi^2}.$$

Final result

Hence, the absolute value of each term of the Fourier series of $f(x)$ is at most equal to

$$|a_0| + \frac{4M\ell^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

which is convergent. Hence that Fourier series converges and the proof is complete.

Piecewise continuous function

A function f is said to be piecewise continuous in $[-\ell, \ell]$ if

- $f(x)$ is defined and continuous in $\forall\, x \in [-\ell, \ell]$ except at finite number of points in $[-\ell, \ell]$.
- At a point $x_0 \in (-\ell, \ell)$, if function is not continuous, then $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist and are finite.
- At the end point of the interval, $\lim_{x \to -\ell^+} f(x)$ and $\lim_{x \to -\ell^-} f(x)$ exist and are finite.

Convergence of Fourier series for piecewise continuous function

A function $f(x)$ can be expressed as a Fourier series

$$\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}\right]$$

in the interval $[-\ell, \ell]$ (where $a_0,\ a_n$ and b_n are constants) provided

- $f(x)$ is periodic (with period 2ℓ), single valued and finite,
- $f(x)$ is piecewise continuous in $[-\ell, \ell]$,
- $f(x)$ has left hand derivative and right hand derivative at each point in the interval.

Theorem

Let $f(x)$ and $f'(x)$ be piecewise continuous functions on the interval $[-\ell, \ell]$. Then the Fourier series of $f(x)$ converges to $f(x)$ at the point of continuity. At the point of discontinuity, say $x_0 \in (-\ell, \ell)$, the Fourier series converges to

$$\frac{1}{2}[f(x_0^+) + f(x_0^-)]$$

where $f(x_0^+)$ and $f(x_0^-)$ are the right and the left hand limits of $f(x)$ at x_0.

At both the end points of the interval $[-\ell, \ell]$, the Fourier series converges to

$$\frac{1}{2}[f(-\ell+) + f(\ell-)]$$

Problem 3

Find the Fourier series expansion of the following function

- $$f(x) = \begin{cases} -\pi & \text{if } -\pi < x < 0, \\ x & \text{if } 0 \leq x < \pi. \end{cases}$$

Hence deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots = \dfrac{\pi^2}{8}$.

Problem 4

Find the Fourier series expansion of the following function

- $$f(x) = \begin{cases} -1 & \text{if } -\pi < t < -\pi/2, \\ 0 & \text{if } -\pi/2 < t < \pi/2, \\ 1 & \text{if } \pi/2 < t < \pi. \end{cases}$$

  Hence determine the value of

$$\frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right).$$

Even and odd function

A function f is said to be

- an even function if $f(-x) = f(x)$.
- an odd function if $f(-x) = -f(x)$.

For example,

$$f(x) = |x|,\ x^2,\ e^{-x^2},\ \cos x \text{ etc.}$$

are even functions and

$$g(x) = x^3,\ x,\ \sin x,\ -\cos x \text{ etc.}$$

are odd functions.

Properties of even and odd function

We know that if $f(x)$ is an even function, then

$$\int_{-\ell}^{\ell} f(x)dx = 2\int_{0}^{\ell} f(x)dx.$$

And, if $f(x)$ is an odd function, then

$$\int_{-\ell}^{\ell} f(x)dx = 0.$$

Fourier series of even function

The Fourier series expansion for an even periodic function $f(x)$ in an interval $[-\ell, \ell]$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$$

where

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)dx \text{ and } a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx,$$

$n = 1, 2, 3, \ldots$

Fourier series of odd function

The Fourier series expansion for an odd periodic function $f(x)$ in an interval $[-\ell, \ell]$ is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx,$$

$n = 1, 2, 3, \ldots$

Problem 5

Given,

$$f(x) = \begin{cases} -x+1 & \text{if } -\pi \leq x \leq 0, \\ x+1 & \text{if } 0 \leq x \leq \pi, \end{cases}$$

with $f(x+2\pi) = f(x)$.

- Is the function even or odd?
- Find the Fourier series for $f(x)$ and hence determine the value of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Problem 6

Find the Fourier series expansion of the periodic function $f(x)$

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0, \\ k & \text{if } 0 < x < \pi. \end{cases}$$

with $f(x + 2\pi) = f(x)$.

- Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

Problem 7

Find the Fourier series expansion of the following function

- $f(x) = 4 - x^2, \ -2 \leq x \leq 2$, with $f(x+4) = f(x)$.

Fourier half range series

Half range series plays an important role in several engineering and physical applications where it is required to get the Fourier series expansion of a function in an interval $(0, \ell)$, ℓ is the half of the period.

Now, it is possible to extend $f(x)$ to the other half $[-\ell, 0]$ of $[-\ell, \ell]$, so that $f(x)$ is either an even or an odd function. In the first case, it is called an even periodic extension of $f(x)$, while in the second case, it is called an odd periodic extension of $f(x)$.

Continued……

- If we do an even periodic extension of $f(x)$, then $f(x)$ is an even function in $[-\ell, \ell]$. Therefore, $f(x)$ has a Fourier cosine series.
- If we do an odd periodic extension of $f(x)$, then $f(x)$ is an odd function in $[-\ell, \ell]$. Therefore, $f(x)$ has a Fourier sine series.

If a function is defined on a half interval $[0, \ell]$, then we can obtain a Fourier cosine or a Fourier sine series expansion, by suitable periodic extensions, depending on the problem.

Fourier sine series

The Fourier sine series expansion of a piecewise continuous function $f(x)$ on the half-range interval $[0, \ell]$ is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

Fourier cosine series

The Fourier cosine series expansion of a piecewise continuous function $f(x)$ on the half-range interval $[0, \ell]$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$$

where

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)dx \text{ and } a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$$

Problem 8

Obtain cosine and sine series for $f(x) = x$ in the interval $0 \leq x \leq \pi$.

Hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}.$$

Problem 9

Expand

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{if } 0 < x < \frac{1}{2}, \\ x - \frac{3}{4} & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

as a Fourier sine series.

Problem 10

Find the Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$.

Hence show that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\infty = \frac{\pi - 2}{4}.$$

Parseval's identity

The Parseval's identity is given as

$$\int_{-\ell}^{\ell} (f(x))^2 dx = \ell \left\{ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\},$$

provided the Fourier series for $f(x)$ converges uniformly in $(-\ell, \ell)$.

Corollary 1

If the half-range cosine series for the function $f(x)$ in $(0, \ell)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right),$$

then the Parseval's formula is given as

$$\int_0^{\ell} (f(x))^2 dx = \frac{\ell}{2}\left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \dots\infty\right)$$

Corollary 2

If the half-range sine series for the function $f(x)$ in $(0, \ell)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right),$$

then the Parseval's formula is given as

$$\int_0^{\ell} (f(x))^2 dx = \frac{\ell}{2}(b_1^2 + b_2^2 + b_3^2 + \ldots\infty)$$

Root mean square value

The root mean square (rms) value of the function $f(x)$ in an interval (a, b) is defined by

$$[f(x)]_{rms} = \sqrt{\left[\frac{\int_a^b [f(x)]^2 dx}{b-a}\right]}$$

It is also known as the effective value of the function.

It has applications in the theory of mechanical vibrations and in electric circuit theory.

Problem 10

Apply Parseval's identity to the function

$$f(x) = x, \ -\pi \leq x \leq \pi, \ f(x + 2\pi) = f(x),$$

and hence deduce that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots = \frac{\pi^2}{6}.$$

Problem 11

Using Parseval's identity to the function

$$f(x) = x^2, \ -\pi \le x \le \pi, \ f(x+2\pi) = f(x),$$

show that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90}.$$

Problem 12

For the function, $f(x) = x + x^2,\ -\pi \le x \le \pi,\ f(x+2\pi) = f(x)$, apply Parseval's identity to evaluate the value of

$$\sum_{n=1}^{\infty}\left(\frac{4}{n^4} + \frac{1}{n^2}\right).$$

Complex form of Fourier Series

The Fourier series of a periodic function $f(x)$ of period 2ℓ, is

$$f(x) = \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}\right], \qquad (1)$$

This can be expressed in complex form, which sometimes makes calculations easier in the problems.

Continued….

We know,

$$e^{i(n\pi x)/\ell} = \cos\frac{n\pi x}{\ell} + i\sin\frac{n\pi x}{\ell}$$

and

$$e^{-i(n\pi x)/\ell} = \cos\frac{n\pi x}{\ell} - i\sin\frac{n\pi x}{\ell}$$

Adding and subtracting the above expressions, we obtain

$$\cos\frac{n\pi x}{\ell} = \frac{1}{2}\left(e^{i(n\pi x)/\ell} + e^{-i(n\pi x)/\ell}\right)$$

$$\text{and}\quad \sin\frac{n\pi x}{\ell} = \frac{1}{2i}\left(e^{i(n\pi x)/\ell} - e^{-i(n\pi x)/\ell}\right)$$

Continued…

Using the fact that $1/i = -i$, we have

$$a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} = \frac{1}{2}a_n\left(e^{i(n\pi x)/\ell} + e^{-i(n\pi x)/\ell}\right) + \frac{1}{2i}b_n\left(e^{i(n\pi x)/\ell} - e^{-i(n\pi x)/\ell}\right)$$

$$= \frac{1}{2}(a_n - ib_n)e^{i(n\pi x)/\ell} + \frac{1}{2}(a_n + ib_n)e^{-i(n\pi x)/\ell}$$

$$= c_n e^{i(n\pi x)/\ell} + c_{-n}e^{-i(n\pi x)/\ell}, \qquad (2)$$

where

$$c_n = \frac{1}{2}(a_n - ib_n) \quad \text{and} \quad c_{-n} = \frac{1}{2}(a_n + ib_n).$$

Continued…

Inserting $\frac{a_0}{2} = c_0$, and using the expression (2) in (1), we obtain

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{i(n\pi x)/\ell} + c_{-n} e^{-i(n\pi x)/\ell}). \quad (3)$$

If f is real, then $c_n = \bar{c}_{-n}$.

Continued

From the Euler's formula, we have

$$c_0 = \frac{a_0}{2} = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx.$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\left(\cos\frac{n\pi x}{\ell} - i\sin\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-i(n\pi x)/\ell}dx$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\left(\cos\frac{n\pi x}{\ell} + i\sin\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{i(n\pi x)/\ell}dx$$

Continued

The expression (3) can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(n\pi x)/\ell},$$

where

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-i(n\pi x)/\ell} dx$$

This is called the complex form of the Fourier series, or, the complex Fourier series of $f(x)$.

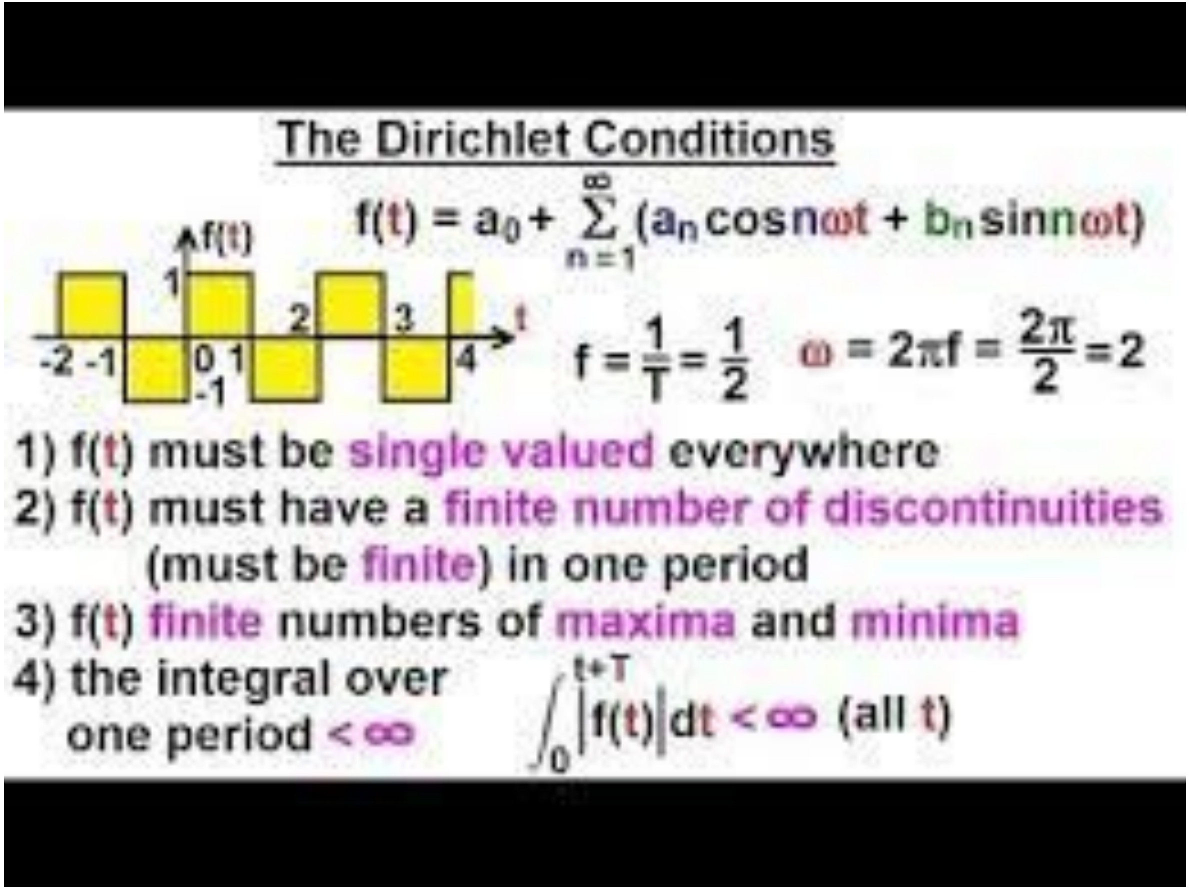
Problem

Find the complex form of the Fourier series of the following functions

- $f(x) = e^{-x}$ in $-\pi < x < \pi$, $f(x + 2\pi) = f(x)$.

- $$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0, \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

$f(x + 2\pi) = f(x)$.


The Dirichlet Conditions
$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$
f(t)
1
-2 -1 0 1 2 3 4
-1
t
$f = \frac{1}{T} = \frac{1}{2}$ $\omega = 2\pi f = \frac{2\pi}{2} = 2$
1) f(t) must be single valued everywhere
2) f(t) must have a finite number of discontinuities
(must be finite) in one period
3) f(t) finite numbers of maxima and minima
4) the integral over one period < ∞
$\int_0^{t+T} |f(t)| dt < \infty$ (all t)

4. Integrals Transforms 10 Lectures

Fourier Transforms: Fourier Integral theorem. Fourier Transform. Examples. Fourier transform of trigonometric, Gaussian, finite wave train & other functions. Representation of Dirac delta function as a Fourier Integral. Fourier transform of derivatives, Inverse Fourier transform, Properties of Fourier transforms (translation, change of scale, complex conjugation, etc.). Three dimensional Fourier transforms with examples. Application of Fourier Transforms to differential equations: One dimensional Wave and Diffusion/Heat Flow Equations